Chaos in disordered nonlinear lattices

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Work in collaboration with
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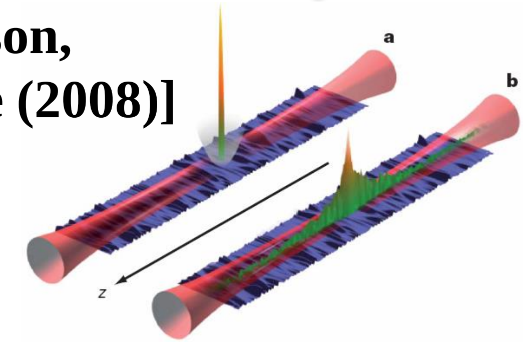
Outline

- **Disordered lattices:**
  - ✓ **The quartic Klein-Gordon (KG) model**
  - ✓ **The disordered nonlinear Schrödinger equation (DNLS)**
  - ✓ **Different dynamical behaviors**
- **Chaotic behavior of the KG model**
  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions**
- **Numerical Integration methods**
- **Summary**

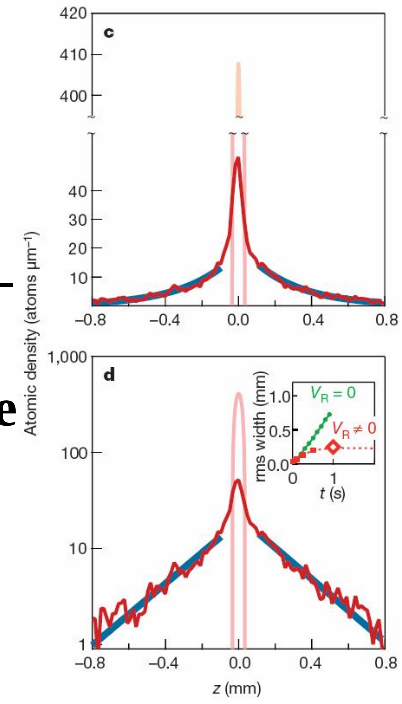
Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]

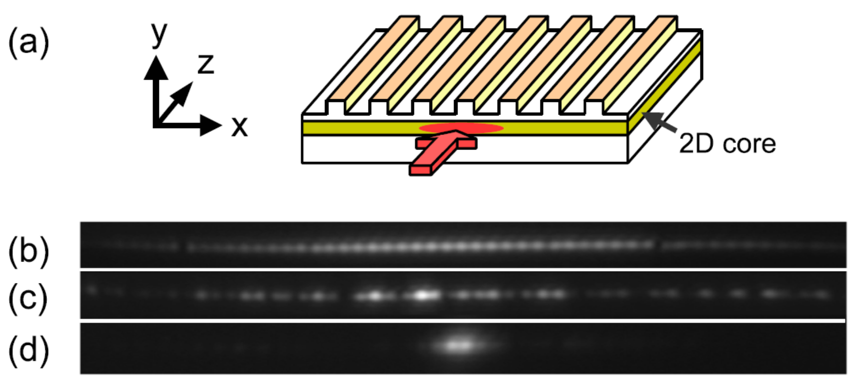


Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]



Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2$$

with **fixed boundary conditions** $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: **W** and the **total energy E.** $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2},\frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l=A_l \exp(i\omega t)$. **Normal modes (NMs)** $A_{\nu,l}$ **- Eigenvalue problem:**

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right)$$

where ε_l **chosen uniformly from** $\left[-\frac{W}{2},\frac{W}{2}\right]$ and β **is the nonlinear parameter.**

Conserved quantities: The energy and the norm $S = \sum_l |\psi_l|^2$ of the wave packet.

Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

$$z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2}\left(\dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2\right)$$

, where A_ν is the amplitude of the νth NM (KG) or norm distributions (DNLS).

Second moment: $$m_2 = \sum_{\nu=1}^{N} (\nu - \bar{\nu})^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$$

Participation number: $$P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$$

measures the number of stronger excited modes in z_ν.

Single mode $P=1$. Equipartition of energy $P=N$.

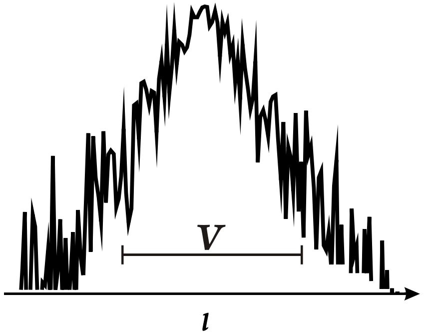
Scales

Linear case: $\omega_\nu^2 \in \left[\frac{1}{2}, \frac{3}{2}+\frac{4}{W}\right]$**, width of the squared frequency spectrum:**

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate: $V \sim \frac{1}{\sum_{l=1}^{N} A_{\nu,l}^4}$



Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta|\psi_l|^2)$$

The relation of the two scales $d_K \le \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes,

δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$

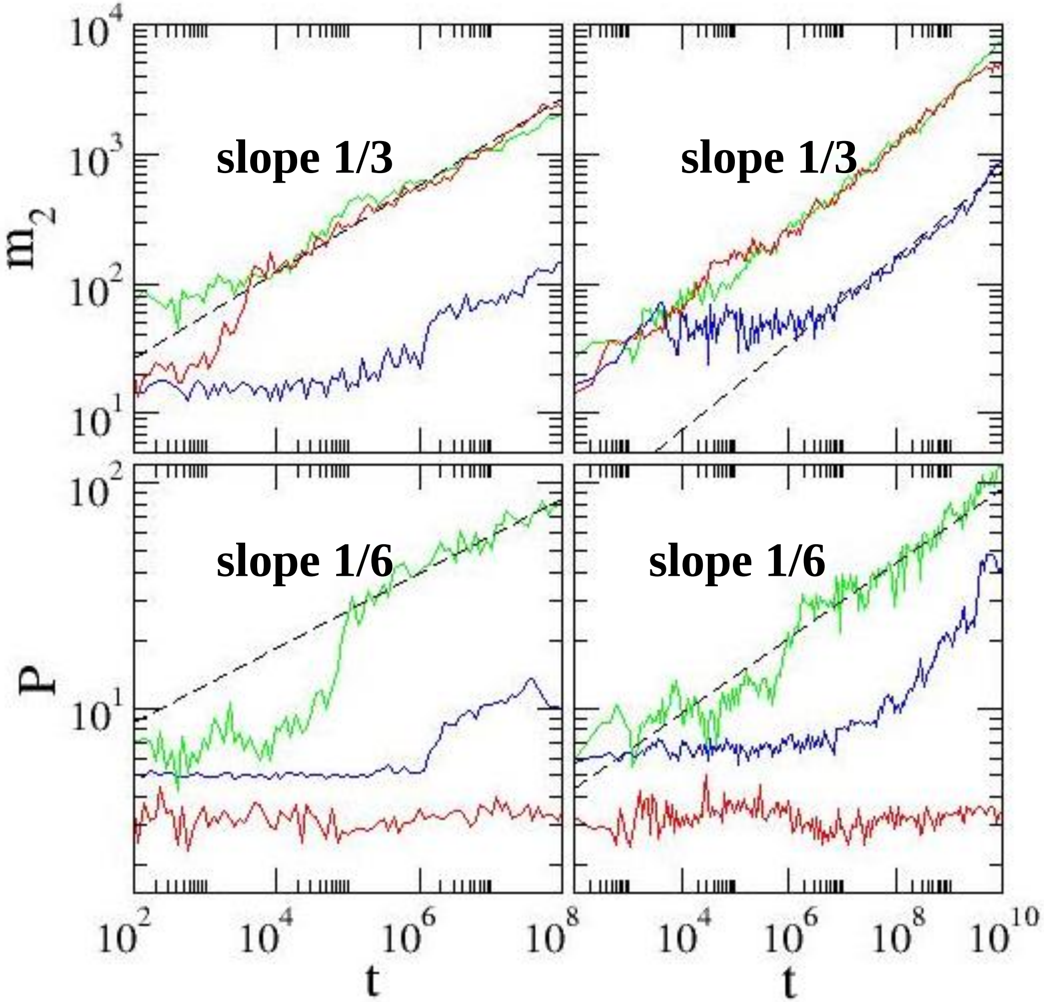
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DNLS **W=4, β= 0.1, 1, 4.5** **KG** **W = 4, E = 0.05, 0.4, 1.5**



No strong chaos regime

In weak chaos regime we averaged the measured exponent α (m_2~t^α) over 20 realizations:

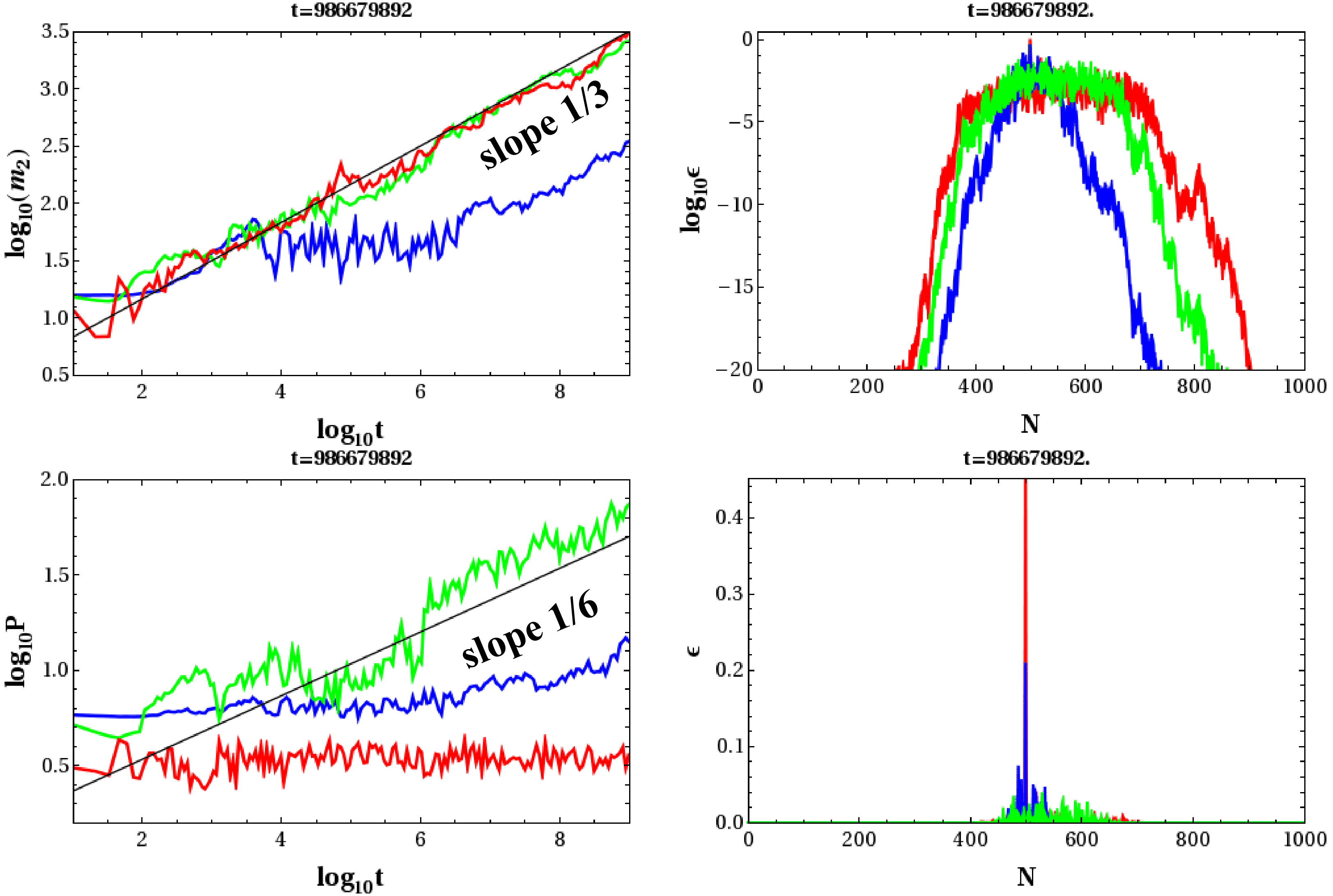
α=0.33±0.05 (KG)

α=0.33±0.02 (DLNS)

Flach et al., PRL (2009)

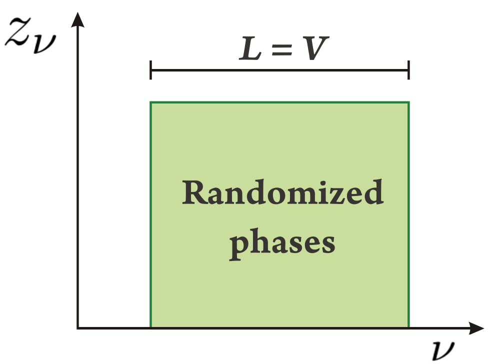
S. et al., PRE (2009)

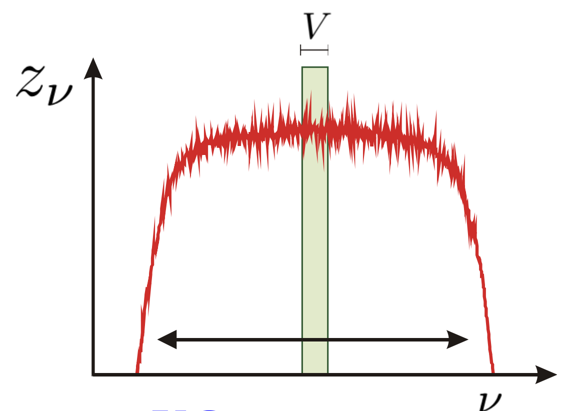
KG: Different spreading regimes



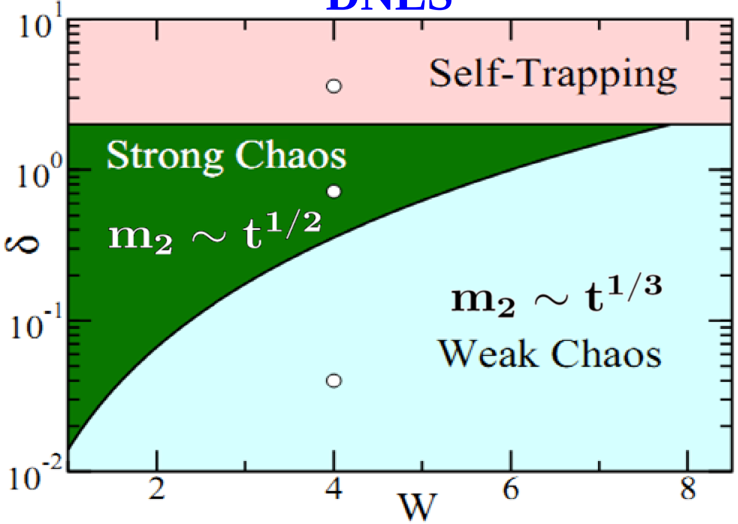
Crossover from strong to weak chaos

We consider compact initial wave packets of width *L=V* [Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)].

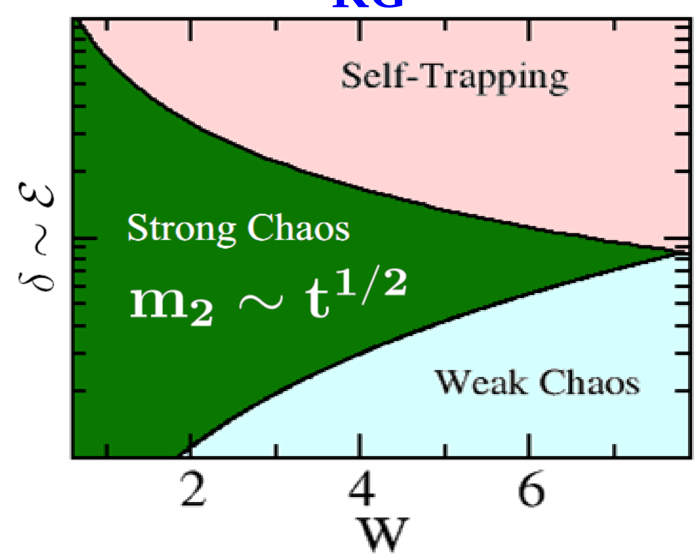




DNLS



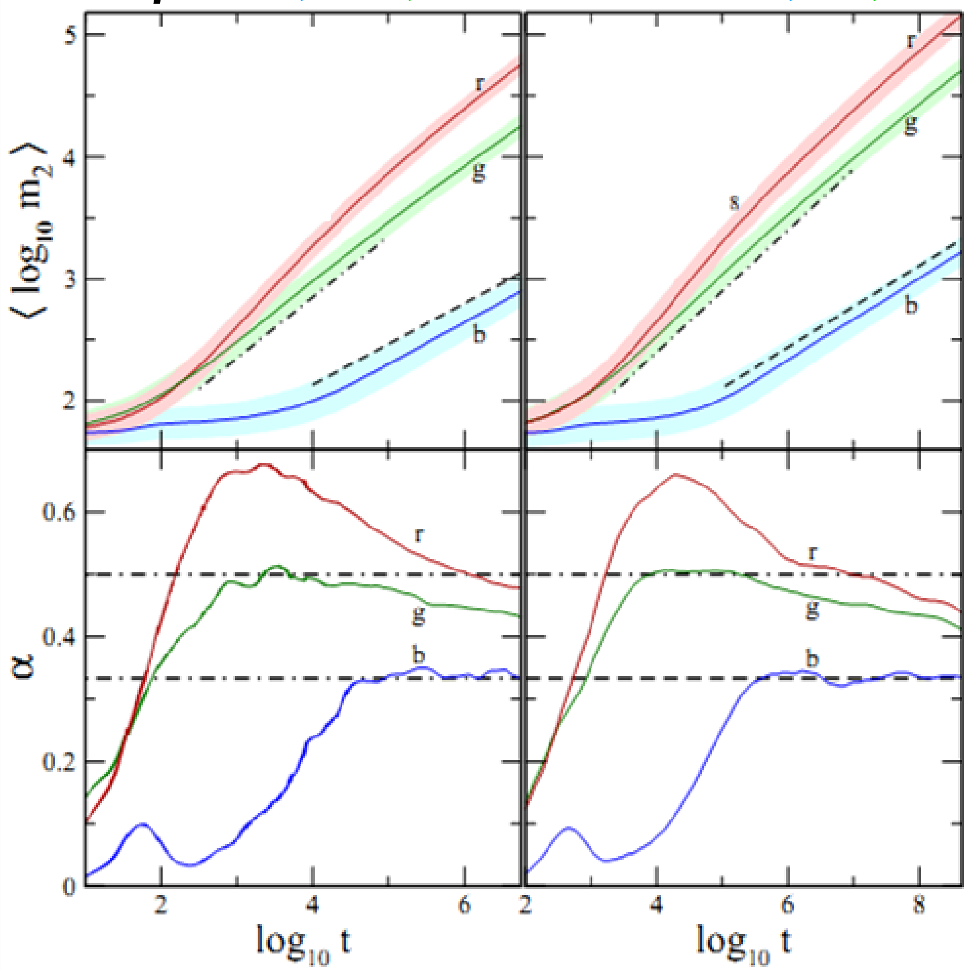
KG



Crossover from strong to weak chaos (block excitations)

DNLS β= 0.04, 0.72, 3.6 **KG E= 0.01, 0.2, 0.75**

W=4



α=1/2

α=1/3

Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

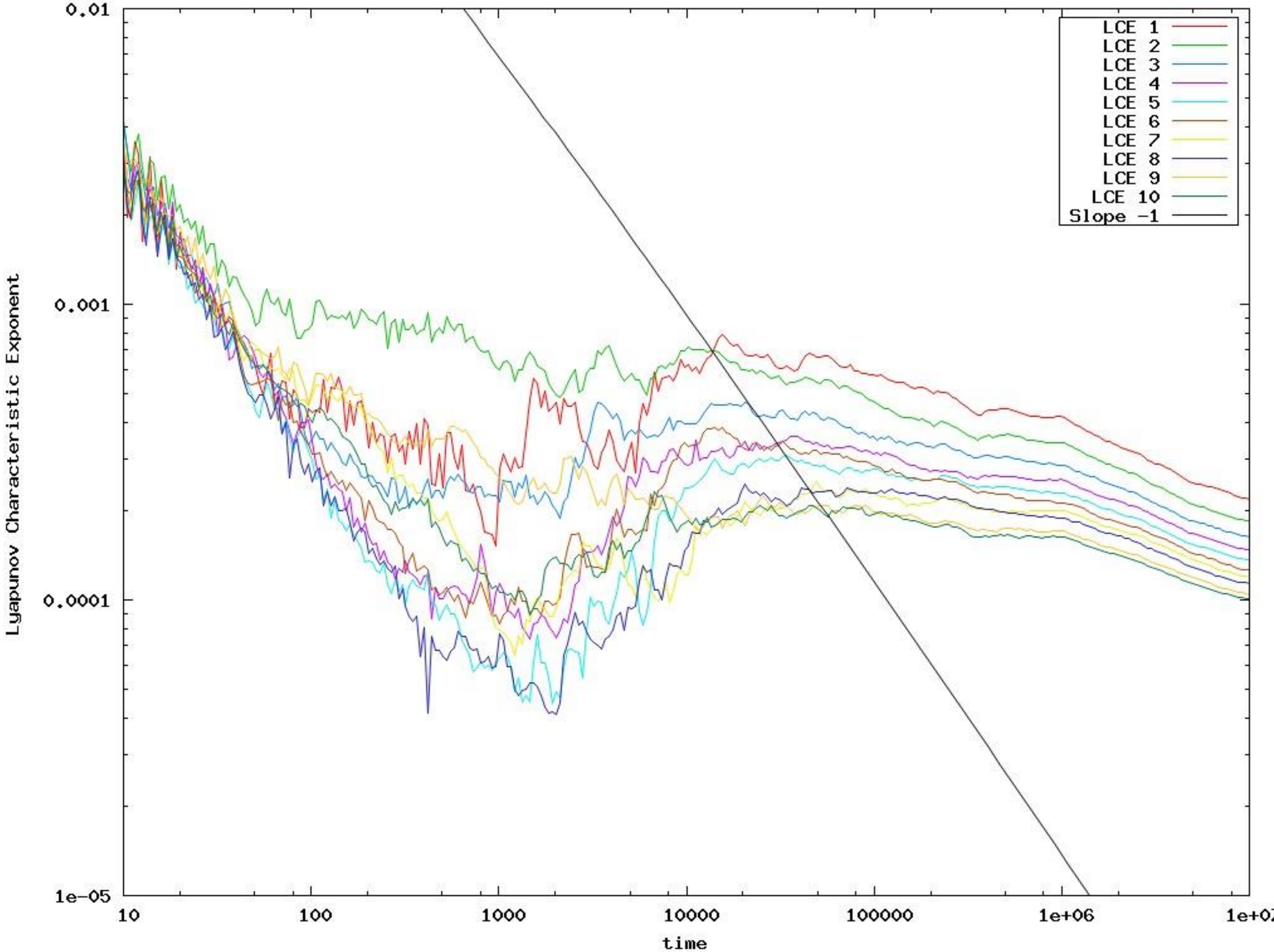
Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\text{mLCE} = \lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\vec{v}(t)\|}{\|\vec{v}(0)\|}$$

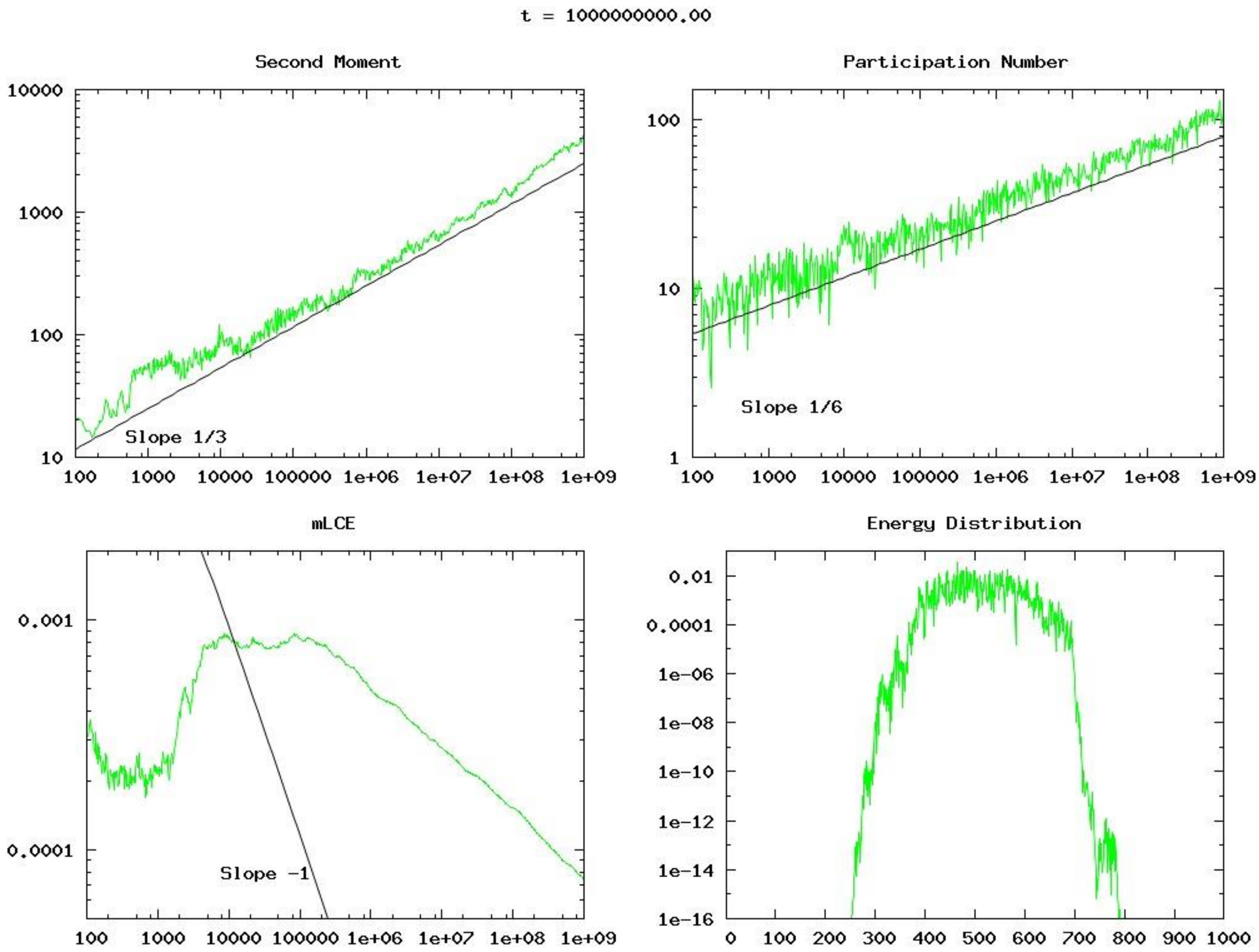
$\lambda_1 = 0 \rightarrow$ **Regular motion** $\propto (t^{-1})$

$\lambda_1 \neq 0 \rightarrow$ **Chaotic motion**

KG: LEs for single site excitations (E=0.4)

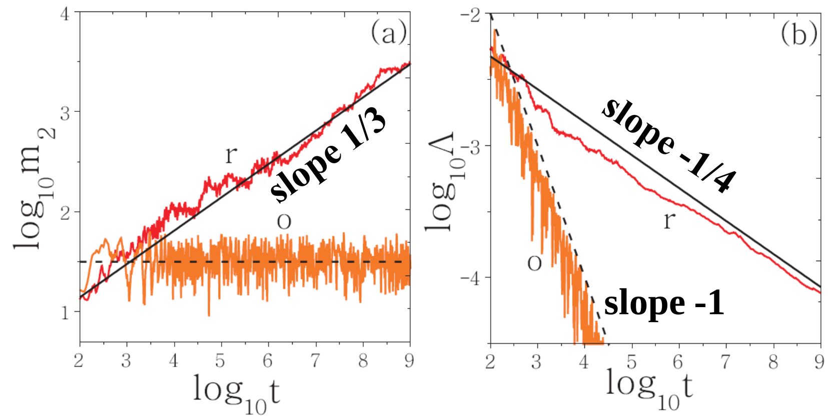


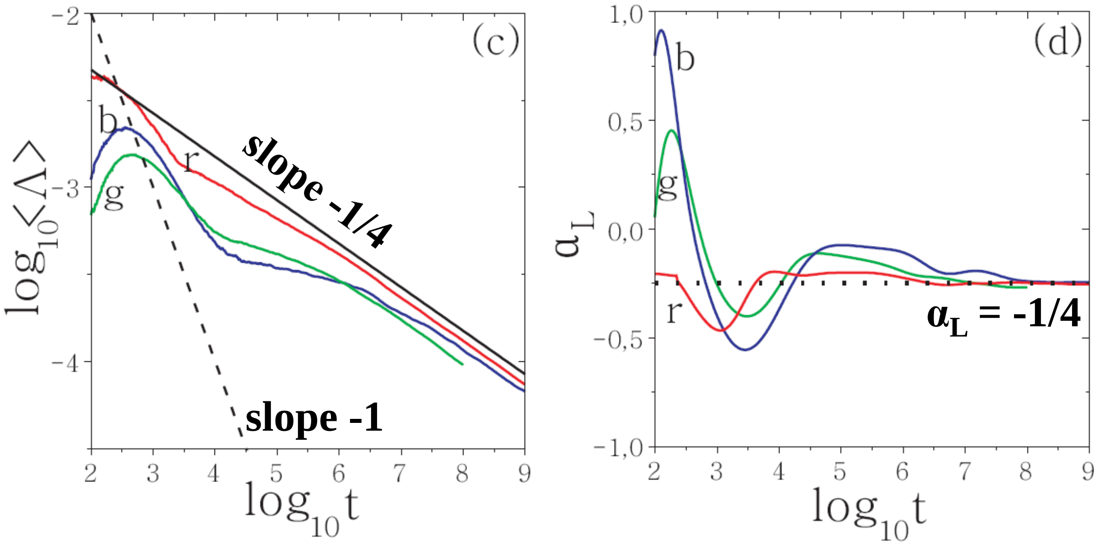
KG: Weak Chaos (E=0.4)



KG: Weak Chaos

Individual runs
Linear case
E=0.4, W=4





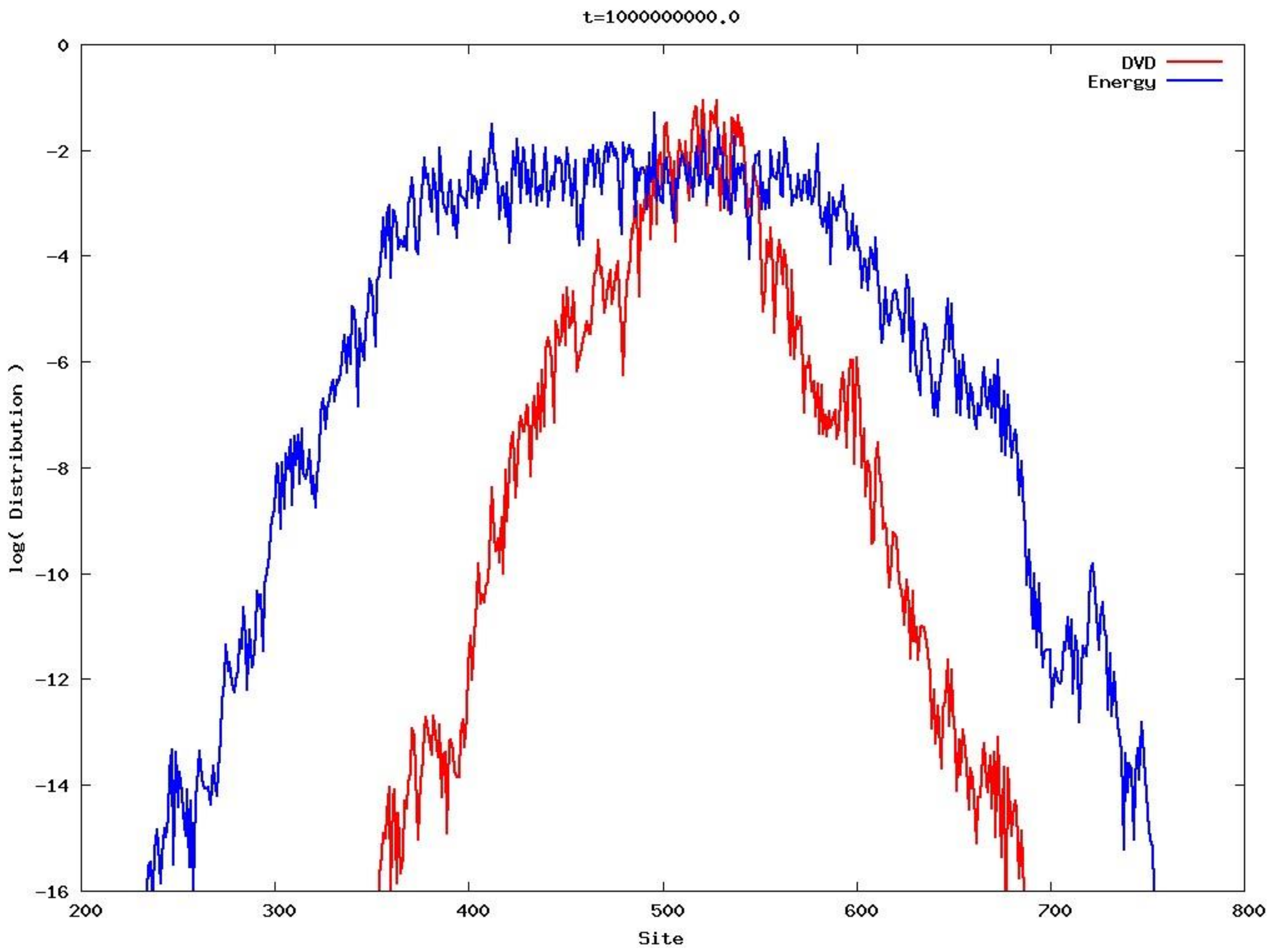
$$\alpha_L = \frac{d\left(\log\langle\Lambda\rangle\right)}{d\log t}$$

Average over 50 realizations

Single site excitation E=0.4, W=4
Block excitation (21 sites) E=0.21, W=4
Block excitation (37 sites) E=0.37, W=3

S. et al. PRL (2013)

Deviation Vector Distributions (DVDs)

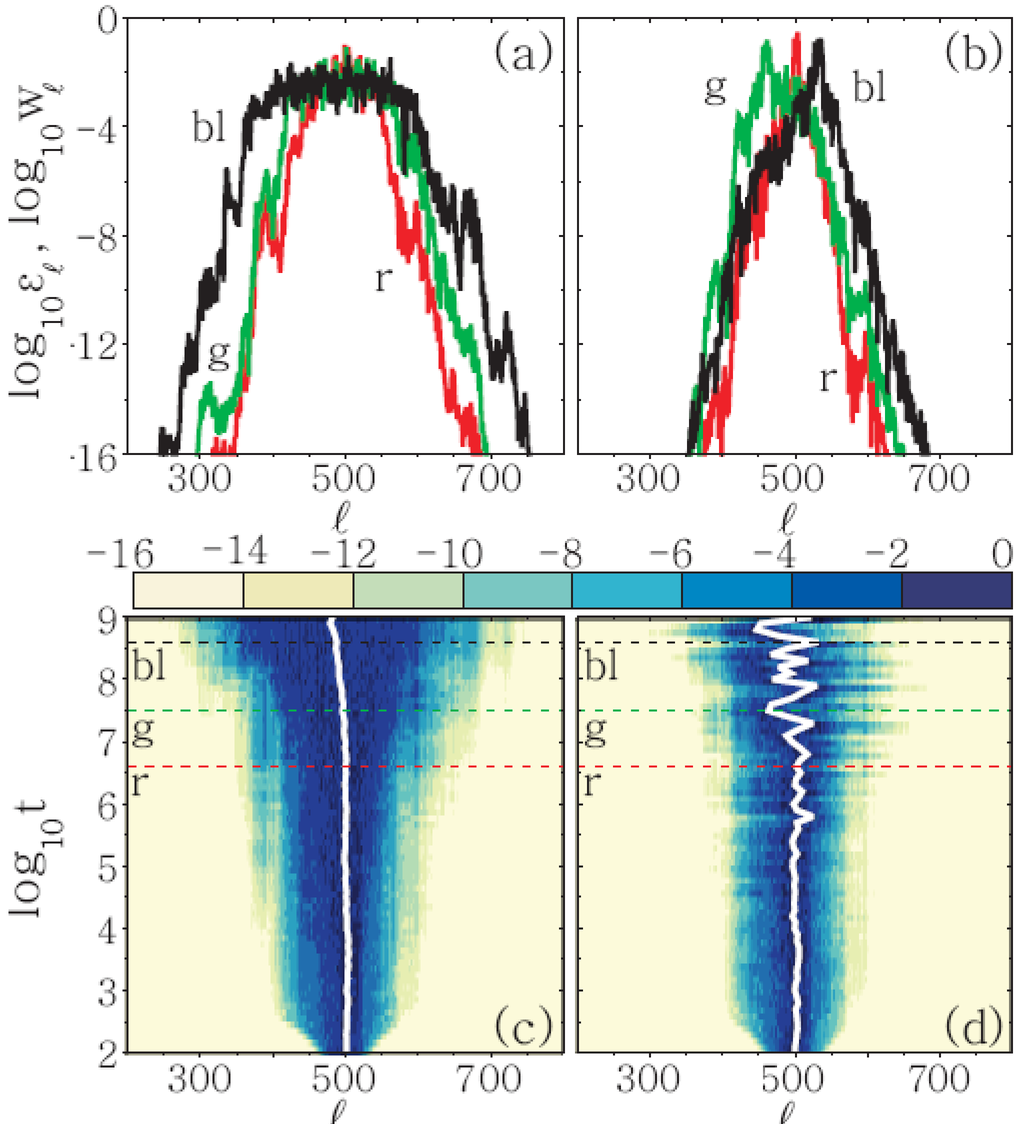


Deviation vector:

v(t)=(δu$_1$(t), δu$_2$(t),…, δu$_N$(t), δp$_1$(t), δp$_2$(t),…, δp$_N$(t))

DVD: $w_l = \dfrac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$

Deviation Vector Distributions (DVDs)



Individual run
E=0.4, W=4

Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

Numerical Integration methods

We use Symplecic Integrators for solving numerically

- **the equations of motion, and**
- **the variational equations (Tangent Map method)**

For more information attend the presentation:

'Efficient integration techniques for the long time simulation of the disordered discrete nonlinear Schrödinger equation'

on Monday 31 October at

'MS7. Dynamics of nonlinear lattices and graphs'

Summary

- **We presented three different dynamical behaviors for wave packet spreading in 1d nonlinear disordered lattices:**
  - ✓ **Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$**
  - ✓ **Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$**
  - ✓ **Selftrapping Regime: $\delta>\Delta$**
- **Lyapunov exponent computations show that:**
  - ✓ **Chaos not only exists, but also persists.**
  - ✓ **Slowing down of chaos does not cross over to regular dynamics.**
  - ✓ **Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.**
- **Our results suggest that Anderson localization is eventually destroyed by nonlinearity, since spreading does not show any sign of slowing down.**
- **We emphasize the use of symplectic integartion schemes for such models (talk on Monday 31 October at 'MS7. Dynamics of nonlinear lattices and graphs').**

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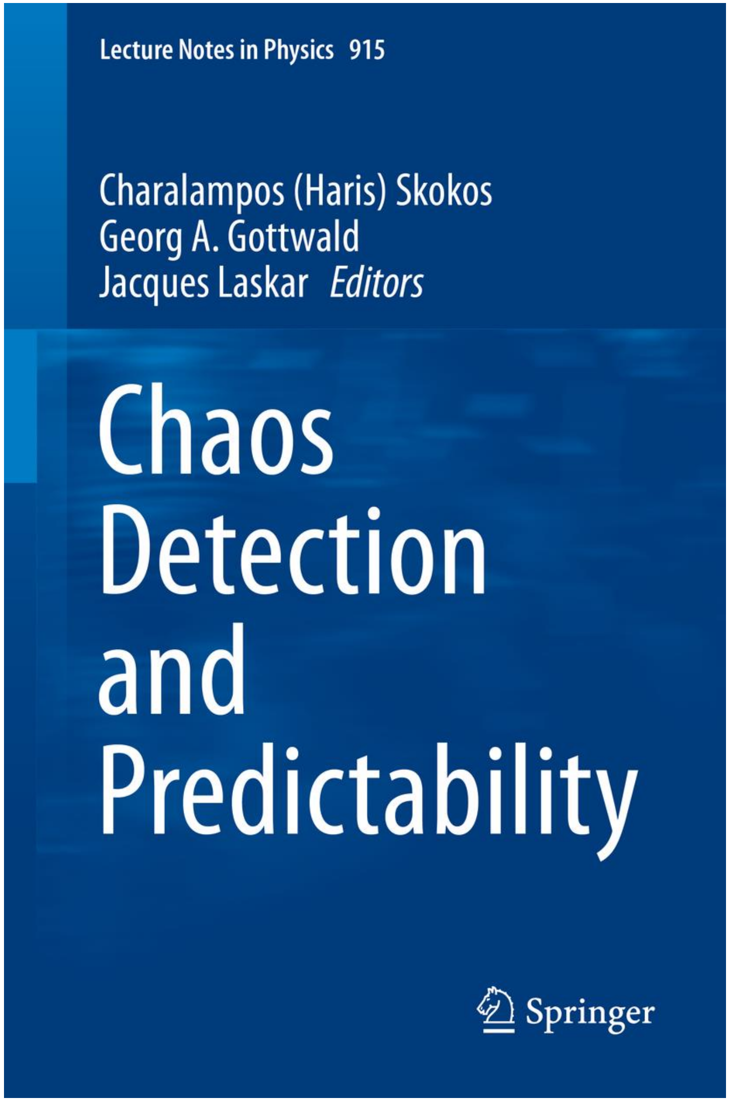
A …shameless promotion



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